Gov 2002 - Causal Inference IV: Repeated Measurements

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- What if selection on the observables doesn't hold, but do have repeated measurements. Can we use this to identify and estimate effects?
- Message: simply having panel data does not identify an effect, but it does allow us to rely on different identifying assumptions.

Basic Idea

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- This is because, even if U_i is unobserved, it is held constant within a unit.
- Thus, by performing analyses within the units, we can control for this unobserved heterogeneity.

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- But estimation is a different issue. Different estimators work differently under either data types.

Fixed effects estimators

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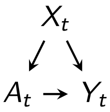

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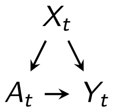

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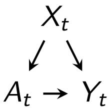

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- History of some variable: $\underline{A}_{it} = (A_1, \ldots, A_t)$.

Basic linear fixed-effects model

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- Key assumptions will be on the relationship between U_i and ε_{it}.
- With no lagged dependent variables in X_{it}, we usually rely on what is called a **strict exogeneity** assumption:

$$E[\varepsilon_{it} | \underline{X}_{iT}, \underline{A}_{iT}, U_i] = 0$$

- This combining this with the above regression, we get the following conditional expectation function:

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- Need to fix one of the unobserved unit effects at $U_1 = 0$ (or fix the mean at 0), $U_2, \ldots, U_N$ are parameters/constants.

Fixed-effects within estimator

- Define the "within" estimator:

$$(Y_{it} - \overline{Y}_i) = (X_{it} - \overline{X}_i)'\beta + \tau(A_{it} - \overline{A}_i) + (\varepsilon_{it} - \overline{\varepsilon}_i)$$

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- This also demonstrates why the assumption of the fixed effects being time-constant is so important.

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- Full rank: $\text{rank}[E[(X_{it} - \overline{X}_i)'(X_{it} - \overline{X}_i)]] = K$

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- More efficient than regular FE when there is serial correlation exists in the errors.

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- Unit-level effects are uncorrelated with treatment and covariates.
- **Important**: implies that ignorability holds without conditioning on U_i, so this is **not** helping us identify causal effects beyond typical regressions.

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- Random effects models gets us consistent standard error estimates.

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  - any other time-varying covariate
- Rules out lagged dependent variables. Why? Informally:
  - Let $Y_{i,t-1} \in X_{it}$.
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Lagged dependent variables

- Strict exogeneity is quite strong.
- Example: economic interdependence between countries ($A_{it} = 1$ if county-dyad i is interdependent in period t) and conflict severity (Y_{it}) between countries.
- Strict exogeneity assumption implies shocks to conflict severity at t uncorrelated with:
  - future values of conflict severity
  - economic interdendence
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  - Implies that ε_{it} uncorrelated with Y_{it}, but this can't be since it is the error for that variable!

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- Dynamic panel literature full of examples of how to use different IV approaches

Heterogeneous treatment effects

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- Get us even closer, note we can do the following:

$$A_i\tau_{it} = A_{it}\tau_c + A_{it}(\tau_{it} - \tau_c)$$

where $\tau_c = E[\tau_{it}]$ is the ATE.

Putting it all together

- Finally, let's assume that the mean of the potential outcome under control is linear:

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- When will $\tau_c = \tau$ from the fixed effects regression models above?

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- Let's make a **strict ignorability** assumption conditional on U_i:

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- Let's find out!

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&= 0
\end{aligned}$$

- Great, the "typical errors" are 0 on average under our strict ignorability assumption.

Non-constant effects errors

- What about the non-constant effects part?

$$E[A_{it}(\tau_{it} - \tau_c)|\underline{X}_{iT}, U_i, A_{it}] = A_{it}(E[\tau_{it} - \tau_c|\underline{X}_{iT}, U_i, A_{it}])$$

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- This is when the treatment effects are independent of the unit effects and the covariates.

Regression bias?

- We've seen this before: it's a general problem with regression and varying treatment effects.

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- Overall ATE still nonparametrically identified, even if the FE regression doesn't estimate it.

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- Estimation here gets more difficult (see Wooldridge, 2002, 11.2)

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- We would just need more assumptions

Basic differences-in-differences model

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- Here we have a period effect, δ_t and a unit effect α_i, and a transitory shock, η_{it}, which has mean zero.

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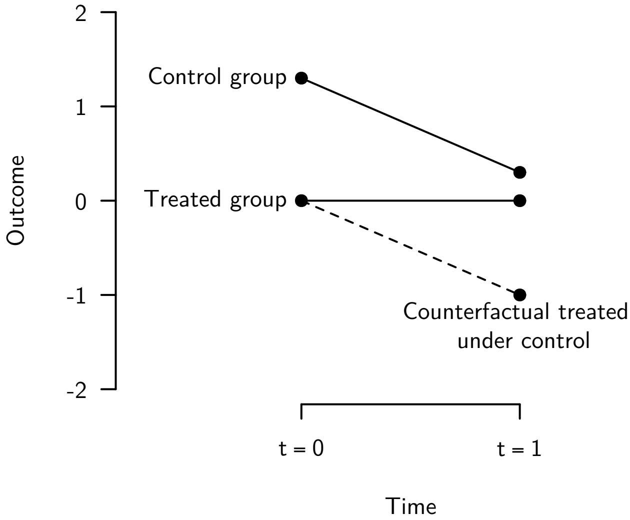
- Specifically, this means treated and control groups have the same trends in the error (on average)

Common trends in a graph



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- Just assumed that control and treatment have the same average secular trends

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- This motivates the differences-in-differences estimator as the difference between these two differences. We can estimate each of these CEFs from the data and compute their sample versions to get an estimate of τ.

Estimation

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- Thus, in the panel data case, we can estimate the effect by regressing the change for each unit, $Y_{i1} - Y_{i0}$, on the treatment.

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- In the Lyall paper, it might be the case that insurgent attacks might be falling in places where there is shelling because rebels attacked in those areas and have moved on.
- Thus, the independence of the treatment and idiosyncratic shocks might only hold conditional on covariates.

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- This approach depends on constant effects and linearity in X_i. Can we generalize?

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  - Synthetic control matching leverages this type of idea

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  - $Y_{i1} = A_i Y_{i1}(1) + (1 - A_i) Y_{i1}(0)$
- We'll focus on two estimands, the ATT,

$$\tau_{ATT} = E[Y_{it}(1) - Y_{it}(0) | A_i = 1]$$

and the conditional ATT:

$$\tau_{ATT}(x) = E[Y_{it}(1) - Y_{it}(0) | X_i = x, A_i = 1]$$

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- Let's make the crucial identifying assumption of a DID model:

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- What does this assumption say? It says that the potential trend under control is the same for the control and treated groups, conditional on covariates.
- Just parallel trends in terms of potential outcomes.
- Note that, if the two groups have the same mean potential outcome under control in the first period,

$$E[Y_{i0}(0)|X_i, A_i = 1] = E[Y_{i0}(0)|X_i, A_i = 0]$$

then this assumption just becomes regular ignorability:

$$E[Y_{i1}(1)|X_i, A_i = 1] = E[Y_{i1}(1)|X_i, A_i = 0]$$

- We can show that this is the key assumption for identifying the DID approach:

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\begin{aligned}
& E[Y_{i1}(1) - Y_{i1}(0) | X_i, A_i = 1] \\
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- Very similar to results above.
- Each CEF could be estimated nonparametrically, but we would run into the curse of dimensionality if X_i is complicated

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- Also, parallel trends assumption may not hold for transformations of the data.
- Nonparametrics will hard with moderately-sized covariate space

Semiparametric estimation with repeated outcomes

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- The tradeoff here is that we have to estimate the propensity score to estimate these weights for each unit:

$$\rho_0(A_i, X_i) = \frac{A_i - \Pr[A_i = 1|X_i]}{\Pr[A_i = 1|X_i](1 - \Pr[A_i = 1|X_i])}$$

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- Sometimes called "inverse probability of treatment weighting" (IPTW)

Proof

- The proof is actually quite straightforward:

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E[\rho_0(Y_{i1} - Y_{i0})|X_i] = \quad & E[\rho_0(A_i, X_i)(Y_{i1} - Y_{i0})|X_i, A_i = 1] \Pr[A_i = 1|X_i] \\
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